

# Constraint on Quantum Gravitational Well and Bose - Einstein Statistics in Noncommutative Space

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## Abstract

In the context of non-relativistic quantum mechanics for the case of both position - position and momentum - momentum noncommuting, the constraint between noncommutative parameters on the quantum gravitational well is investigated. The related topic of guaranteeing Bose - Einstein statistics in the general case are elucidated: Bose - Einstein statistics is guaranteed by the deformed Heisenberg - Weyl algebra itself, independent of dynamics. A special feature of a dynamical system is represented by a constraint between noncommutative parameters. The general feature of the constraint for any system is a direct proportionality between noncommutative parameters with a coefficient depending on characteristic parameters of the system under study. The constraint on the quantum gravitational well is determined up to an arbitrary dimensionless constant.

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Physics in noncommutative space [1–3] has been extensively investigated in literature. This is motivated by studies of the low energy effective theory of D-brane with a nonzero Neveu - Schwarz  $B$  field background. Furthermore, there is some argument that spatial noncommutativity may arise as a quantum effect of gravity. Effects of spatial noncommutativity are apparent only near the string scale, thus we need to work at a level of noncommutative quantum field theory. But based on the incomplete decoupling mechanism one expects that quantum mechanics in noncommutative space (NCQM) may clarify some low energy phenomenological consequences, and lead to qualitative understanding of effects of spatial noncommutativity. In literature NCQM and its applications [4–15] have been studied in detail. But some important questions, such as the guarantee of Bose - Einstein statistics in the general case, have not been resolved.

Recently the quantum gravitational well has attracted attentions [16, 17, 18, 9]. The existence of quantum states of particles in the gravitational field, as ones in electromagnetic and strong fields, is expected for a long time. The lowest stationary quantum state of neutrons in the Earth's gravitational field is identified in the laboratory. In the case of noncommutative space it is noticed that, because of the speciality of its linear potential [9], the constraint between noncommutative parameters on the quantum gravitational well is not clear.

In this paper the discussions are restricted in the context of non-relativistic quantum mechanics. In this paper our attention focuses on elucidating this topic, which is closely related to the above question about guaranteeing Bose - Einstein statistics in the general case. We find that Bose - Einstein statistics is guaranteed by the deformed Heisenberg - Weyl algebra itself, independent of dynamics. A special feature of a dynamical system is represented by a constraint between noncommutative parameters. The general feature of the constraint for any system is a direct proportionality between noncommutative parameters with a coefficient depending on characteristic parameters of the system under study. Such a constraint for the quantum gravitational well is fixed up to a dimensionless constant.

**The Deformed Heisenberg - Weyl Algebra** - In the following we review the background first. In order to develop the NCQM formulation we need to specify the phase space

and the Hilbert space on which operators act. The Hilbert space is consistently taken to be exactly the same as the Hilbert space of the corresponding commutative system [4]. There are different types of noncommutative theories, for example, see a review paper [3].

As for the phase space we consider both position - position noncommutativity (space-time noncommutativity is not considered) and momentum - momentum noncommutativity. In this case the consistent deformed Heisenberg - Weyl algebra [11] is:

$$[\hat{x}_i, \hat{x}_j] = ib\theta\epsilon_{ij}, \quad [\hat{p}_i, \hat{p}_j] = ib\eta\epsilon_{ij}, \quad [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}, \quad (i, j = 1, 2), \quad (1)$$

where  $\theta$  and  $\eta$  are constant parameters, independent of the position and momentum. Here we consider the intrinsic momentum - momentum noncommutativity. It means that the parameter  $\eta$ , like the parameter  $\theta$ , should be extremely small. This is guaranteed by a direct proportionality provided by a constraint between them (See below). The  $\epsilon_{ij}$  is a two-dimensional antisymmetric unit tensor,  $\epsilon_{12} = -\epsilon_{21} = 1$ ,  $\epsilon_{11} = \epsilon_{22} = 0$ . In Eq. (1)  $b$  is a dimensionless constant, which will be fixed later.

In literature the deformed Heisenberg - Weyl algebra (1) has different realizations by undeformed variables [6, 9]. Here we show that representations of deformed variables  $\hat{x}_i$  and  $\hat{p}_i$  by undeformed variables  $x_i$  and  $p_i$  are *uniquely* fixed by a requirement of consistency of the framework. We consider the following ansatz (Henceforth summation convention is used)

$$\hat{x}_i = \xi(x_i - \frac{1}{2\hbar}\theta\epsilon_{ij}p_j), \quad \hat{p}_i = \xi(p_i + \frac{1}{2\hbar}\eta\epsilon_{ij}x_j), \quad (2)$$

where  $x_i$  and  $p_i$  satisfy the undeformed Heisenberg - Weyl algebra

$$[x_i, x_j] = [p_i, p_j] = 0, \quad [x_i, p_j] = i\hbar\delta_{ij}.$$

In Eq. (2) parameter  $\xi$  is a dimensionless constant, which can be fixed as follows. Inserting Eqs. (2) into the first equation of Eqs. (1), it follows that  $[\hat{x}_i, \hat{x}_j] = ib\theta\epsilon_{ij} = i\xi^2\theta\epsilon_{ij}$ , thus  $b = \xi^2$  (For the equation  $[\hat{p}_i, \hat{p}_j] = ib\eta\epsilon_{ij}$ , we obtain the same result). Furthermore, the Heisenberg commutation relation  $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$  should be maintained by Eqs. (2). Inserting Eqs. (2) into the third equation of Eqs. (1), we obtain that  $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij} = i\hbar\xi^2(1 + \theta\eta/4\hbar^2)\delta_{ij}$ . Thus both parameters  $b$  and  $\xi$  are fixed:

$$\xi = (1 + \theta\eta/4\hbar^2)^{-1/2}, \quad b = \xi^2. \quad (3)$$

The parameter  $\xi$  is called the scaling factor. When  $\eta = 0$ , we have  $\xi = 1$ . The deformed Heisenberg - Weyl algebra (1) reduces to the one of only position - position noncommuting. For the case of both position - position and momentum - momentum noncommuting the scaling factor  $\xi$  plays a role for guaranteeing consistency of the framework.

Another choice of the scaling factor leads to the relations obtained in Ref. [9] with the Planck constant  $\hbar$  replaced by an effective Planck constant  $\hbar_{eff} = \hbar(1 + \theta\eta/4\hbar^2)$ , which are consistent with the same algebra (1).

It is worth noting [14] that, unlike the case of only position - position noncommuting, the determinant  $\mathcal{R}_s$  of the transformation matrix  $R_s$  between  $(\hat{x}_1, \hat{x}_2, \hat{p}_1, \hat{p}_2)$  and  $(x_1, x_2, p_1, p_2)$  in Eqs. (2) is  $\mathcal{R}_s = \xi^4(1 - \theta\eta/4\hbar^2)^2$ . When  $\theta\eta = 4\hbar^2$ , the matrix  $R_s$  is singular. This means that the deformed variables  $(\hat{x}_i, \hat{p}_i)$  and the undeformed variables  $(x_i, p_i)$  are not completely equivalent in physics.

**The Quantum Gravitational Well** - Recently, the lowest stationary quantum state of a particle in the Earth's gravitational field is identified in the laboratory [16–18]. The experimental setup is as follows. A system of a particle of mass  $\mu_n$  in the potential well formed by the two dimensional constant Earth's gravitational field,  $\mathbf{g} = -g\mathbf{e}_x$ , where  $g$  is the standard gravitational acceleration at the sea level, and a horizontal mirror placed at  $x = 0$ . This system is known as the quantum gravitational well. In the laboratory the gravitational field alone does not create a potential well, as it can only confine particles to fall along gravity lines. Thus a horizontal mirror is necessary for creating the well. In order to avoid that electromagnetic effects overlap the effect of the Earth's gravitational field, in the experiment some neutral particles with a long lifetime, such as slow neutrons, are chosen.

If NCQM is a realistic physics, low energy quantum phenomena should be reformulated in terms of the deformed operators. The Hamiltonian of the quantum gravitational well, formulated in terms of the deformed phase space variables  $\hat{x}_i$  and  $\hat{p}_i$ , is

$$\hat{H} = \frac{1}{2\mu_n}\hat{p}_i^2 + \mu_n g \hat{x}_1. \quad (4)$$

This system exhibits both theoretical and experimental interests. Investigations of low energy properties of this system may help to explore effects of spatial noncommutativity,

and hopefully shed some new light on physical reality. In many systems, the potential can be modelled by a harmonic oscillator through an expansion about its minimum. The speciality of the gravitational well is that its linear potential is not the case. Recently it is argued [9] that, because of its speciality, a direct proportionality between noncommutative parameters in the constrained condition explored in two dimensional harmonic oscillator [11] may not apply to the quantum gravitational well.

This problem is related to guarantee Bose - Einstein statistics in the case of both position - position and momentum - momentum noncommuting. We find that the maintenance of Bose - Einstein statistics is determined by the deformed Heisenberg - Weyl algebra itself, independent of dynamics. A special feature of a dynamical system is represented by a constraint between noncommutative parameter, which can be determined by the deformed bosonic algebra. The general feature of such a constraint is a direct proportionality between noncommutative parameters  $\eta$  and  $\theta$ , which works for any dynamical systems, including the quantum gravitational well.

In the following we first investigate Bose - Einstein statistics for the general case.

**The Guarantee of Bose - Einstein Statistics** - In literature the guarantee of noncommutative Bose - Einstein statistics in the case of both position - position and momentum - momentum noncommuting has not been resolved.

In this paragraph we demonstrate that for the general case noncommutative Bose - Einstein Statistics is guaranteed by the following existence theorem:

**Theorem** *Bose - Einstein statistics for the case of both position - position noncommutativity and momentum - momentum noncommutativity is guaranteed by the deformed Heisenberg - Weyl algebra itself, independent of dynamics. The deformed bosonic algebra constitutes a complete and closed algebra.*

In the context of non-relativistic quantum mechanics the proof of this theorem includes two aspects. The first aspect is to construct the general representations of deformed annihilation and creation operators from the deformed Heisenberg - Weyl algebra itself, and to find the complete and closed deformed bosonic algebra. The second aspect is about the general construction of the Fock space of identical bosons in noncommutative space in which the formalism of the deformed bosonic symmetry of restricting the states under the

permutation of identical particles in multi - particle systems can be developed.

On the level of quantum field theory the annihilation and creation operators appear in the expansion of the field operator. In the context of non-relativistic quantum mechanics the deformed annihilation operator  $\hat{a}_i$  can be generally represented by  $\hat{x}_i$  and  $\hat{p}_i$  as

$$\hat{a}_i = c_1(\hat{x}_i + ic_2\hat{p}_i), \quad (5)$$

where the constants  $c_1$  and  $c_2$  can be fixed as follows. The operators  $\hat{a}_i$  and  $\hat{a}_i^\dagger$  should satisfy the bosonic commutation relations  $[\hat{a}_1, \hat{a}_1^\dagger] = [\hat{a}_2, \hat{a}_2^\dagger] = 1$  (to keep the physical meaning of  $\hat{a}_i$  and  $\hat{a}_i^\dagger$ ). From this requirement and the deformed Heisenberg - Weyl algebra (1) it follows that  $c_1 = \sqrt{1/2\hbar c_2}$ . When the state vector space of identical bosons is constructed by generalizing one - particle quantum mechanics, Bose - Einstein statistics should be maintained at the deformed level described by  $\hat{a}_i$ , thus operators  $\hat{a}_i$  and  $\hat{a}_j$  should be commuting:  $[\hat{a}_i, \hat{a}_j] = 0$ . From this equation and the deformed Heisenberg - Weyl algebra (1) it follows that  $ic_1^2\xi^2\epsilon_{ij}(\theta - c_2^2\eta) = 0$ . Thus *the condition of guaranteeing Bose - Einstein statistics* reads

$$c_2 = \sqrt{\theta/\eta}. \quad (6)$$

The general representations of the deformed annihilation and creation operators  $\hat{a}_i$  and  $\hat{a}_i^\dagger$  are

$$\hat{a}_i = \sqrt{\frac{1}{2\hbar}\sqrt{\frac{\eta}{\theta}}} \left( \hat{x}_i + i\sqrt{\frac{\theta}{\eta}}\hat{p}_i \right), \hat{a}_i^\dagger = \sqrt{\frac{1}{2\hbar}\sqrt{\frac{\eta}{\theta}}} \left( \hat{x}_i - i\sqrt{\frac{\theta}{\eta}}\hat{p}_i \right), \quad (7)$$

From Eqs. (7) and (6) it follows that the deformed bosonic algebra of  $\hat{a}_i$  and  $\hat{a}_j^\dagger$  reads

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij} + \frac{i}{\hbar}\xi^2\sqrt{\theta\eta}\epsilon_{ij}, [\hat{a}_i, \hat{a}_j] = 0, (i, j = 1, 2). \quad (8)$$

In Eqs. (8) the three equations  $[\hat{a}_1, \hat{a}_1^\dagger] = [\hat{a}_2, \hat{a}_2^\dagger] = 1$ ,  $[\hat{a}_1, \hat{a}_2] = 0$  are the same as the undeformed bosonic algebra in commutative space; The equation

$$[\hat{a}_1, \hat{a}_2^\dagger] = \frac{i}{\hbar}\xi^2\sqrt{\theta\eta} \quad (9)$$

is a new type. Eqs. (8) constitute a complete and closed deformed bosonic algebra.

The second aspect is about the general construction of the Fock space of identical bosons in noncommutative space. Following the standard procedure of constructing the

Fock space of many - particle systems in commutative space, we shall take Eqs. (8) as the defining relations for the complete and closed deformed bosonic algebra without making further reference to its  $\hat{x}_i, \hat{p}_i$  representations, generalize it to many - particle systems and complete the deformed Bosonic symmetry. In the case of both position - position and momentum - momentum noncommuting the special future is when  $[\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0$  are satisfied, Bose - Einstein statistics is not yet guaranteed. The reason is as follows. Because the new bosonic commutation relation (9) correlates different degrees of freedom, the number operators  $\hat{N}_1 = \hat{a}_1^\dagger \hat{a}_1$  and  $\hat{N}_2 = \hat{a}_2^\dagger \hat{a}_2$  do not commute,  $[\hat{N}_1, \hat{N}_2] \neq 0$ . They have not common eigenstates. In this case the construction of the Fock space is involved.

<sup>1</sup> A full investigations of noncommutative Bose - Einstein statistics should complete the deformed Bosonic symmetry under the permutation of identical particles. Thus we should successfully construct the Fock space of identical bosons.

We introduce the following tilde annihilation and creation operators

$$\tilde{a}_1 = \frac{1}{\sqrt{2\alpha_1}} (\hat{a}_1 + i\hat{a}_2), \quad \tilde{a}_2 = \frac{1}{\sqrt{2\alpha_2}} (\hat{a}_1 - i\hat{a}_2), \quad (10)$$

where  $\alpha_{1,2} = 1 \pm \frac{i}{\hbar} \xi^2 \sqrt{\theta\eta}$ . From Eqs. (8) it follows that the commutation relations of  $\tilde{a}_i$  and  $\tilde{a}_j^\dagger$  read  $[\tilde{a}_i, \tilde{a}_j^\dagger] = \delta_{ij}$ ,  $[\tilde{a}_i, \tilde{a}_j] = [\tilde{a}_i^\dagger, \tilde{a}_j^\dagger] = 0$ ,  $(i, j = 1, 2)$ . Thus  $\tilde{a}_i$  and  $\tilde{a}_i^\dagger$  are explained as the deformed annihilation and creation operators in the tilde system. The tilde number operators  $\tilde{N}_1 = \tilde{a}_1^\dagger \tilde{a}_1$  and  $\tilde{N}_2 = \tilde{a}_2^\dagger \tilde{a}_2$  commute each other,  $[\tilde{N}_1, \tilde{N}_2] = 0$ . The state  $|0, 0\rangle$  is also the vacuum state in the tilde system. A general tilde state  $|\widetilde{m, n}\rangle \equiv$

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<sup>1</sup> The definition of the vacuum state  $|0, 0\rangle$  is the same as in commutative space,  $\hat{a}_i|0, 0\rangle = 0$ ,  $(i = 1, 2)$ . If a general hat state  $|\widehat{m, n}\rangle$  is defined as  $|\widehat{m, n}\rangle \equiv c(\hat{a}_1^\dagger)^m (\hat{a}_2^\dagger)^n |0, 0\rangle$  where  $c$  is the normalization constant, these states  $|\widehat{m, n}\rangle$  are not the eigenstate of  $\hat{N}_1$  and  $\hat{N}_2$ :

$$\hat{N}_1 |\widehat{m, n}\rangle = m |\widehat{m, n}\rangle + \frac{i}{\hbar} m \xi^2 \sqrt{\theta\eta} |m+1, n-1\rangle,$$

$$\hat{N}_2 |\widehat{m, n}\rangle = n |\widehat{m, n}\rangle + \frac{i}{\hbar} n \xi^2 \sqrt{\theta\eta} |m-1, n+1\rangle.$$

Because of the new bosonic commutation relation (9), in calculations of the above equations we should take care of the ordering in the state  $|\widehat{m, n}\rangle$ . The states  $|\widehat{m, n}\rangle$  are not orthogonal each other. For example, the inner product between  $|\widehat{1, 0}\rangle$  and  $|\widehat{0, 1}\rangle$  is:

$$\langle \widehat{1, 0} | \widehat{0, 1} \rangle = -\frac{i}{\hbar} \xi^2 \sqrt{\theta\eta}.$$

Thus  $\{|\widehat{m, n}\rangle\}$  do not constitute an orthogonal complete basis.

$(m!n!)^{-1/2}(\tilde{a}_1^\dagger)^m(\tilde{a}_2^\dagger)^n|0,0\rangle$  are the common eigenstate of  $\tilde{N}_1$  and  $\tilde{N}_2$ :  $\tilde{N}_1|\widetilde{m,n}\rangle = m|\widetilde{m,n}\rangle$ ,  $\tilde{N}_2|\widetilde{m,n}\rangle = n|\widetilde{m,n}\rangle$ , ( $m, n = 0, 1, 2, \dots$ ). We obtain  $\langle\widetilde{m'},n'|\widetilde{m,n}\rangle = \delta_{m'm}\delta_{n'n}$ . Thus  $\{|\widetilde{m,n}\rangle\}$  constitute an orthogonal normalized complete basis of the tilde Fock space.

In the above we proved the theorem. Now we investigate some issues related to the theorem.

*The correlated bosonic commutation relation* - Different from the case in commutative space, the new bosonic commutation relation (9) correlates different degrees of freedom to each other, so it is called the correlated bosonic commutation relation. It encodes effects of spatial noncommutativity at the level of  $\hat{a}_i$  and  $\hat{a}_i^\dagger$ , and plays essential roles in dynamics. It is the origin of the fractional angular momentum [11].

The correlated bosonic commutation relation (9) is consistent with *all* principles of quantum mechanics and Bose - Einstein statistics.

In literature one is not aware of the correlated bosonic commutation relation (9). This means that in literature the deformed bosonic algebra is closed, but not complete.

*Consistency of the framework* - In the above we prove that for the case of both position - position and momentum - momentum noncommuting, Bose - Einstein statistics is guaranteed by the general construction of deformed annihilation and creation operators (7). If momentum - momentum is commuting ( $\eta = 0$ ), it is impossible to obtain  $[\hat{a}_i, \hat{a}_j] = 0$ . We conclude that in order to maintain Bose - Einstein statistics for identical bosons at the deformed level we should consider both position - position noncommutativity and momentum - momentum noncommutativity.

*The Constrained Condition* - The structure of the deformed annihilation and creation operators  $\hat{a}_i$  and  $\hat{a}_i^\dagger$  in Eqs. (7) are determined by the deformed Heisenberg - Weyl algebra (1), independent of dynamics. The special feature of a dynamical system is encoded in the dependence of the factor  $\sqrt{\theta/\eta}$  on characteristic parameters of the system under study. This put a constraint on  $\theta$  and  $\eta$ . From Eq. (6) it follows the following constrained condition

$$\eta = K\theta, \quad (11)$$

where the coefficient  $K = c_2^{-2}$  is a constant with a dimension  $(mass/time)^2$ , and depends on characteristic parameters of the system under study. Thus the momentum - momentum



noncommutative parameter  $\eta$  also depends on characteristic parameters of a dynamical system. Such a dependency of  $\eta$  on parameters of the Hamiltonian (or the action) can be understood based on the following observation: noncommutativity between momenta arises naturally as a consequence of noncommutativity between coordinates, as momenta are defined to be the partial derivatives of the action with respect to the noncommutative coordinates [19].

Eq. (11) shows that the general feature of such a constraint is a direct proportionality between noncommutative parameters  $\eta$  and  $\theta$ . The deformed Heisenberg - Weyl algebra (1) is foundations of noncommutative quantum theories. A result derived from this algebra is a fundamental one. The condition (6) of guaranteeing Bose - Einstein statistics is determined by the deformed Heisenberg - Weyl algebra. This means that Eq. (11) is based on a fundamental principle in noncommutative quantum theories. It is a general result, and can apply to any dynamical systems.

*Realizations of the deformed annihilation and creation operators by the undeformed ones* - The representations of the deformed annihilation and creation operators  $\hat{a}_i$  and  $\hat{a}_i^\dagger$  by the undeformed ones  $a_i$  and  $a_i^\dagger$  are consistently determined by Eqs. (1)- (3) and (5)-(7) as follows.

By the same procedure leading to Eqs. (7), we obtain the general representation of the undeformed annihilation operator

$$a_i = c'_1(x_i + ic'_2 p_i), \quad c'_1 = \sqrt{\frac{1}{2\hbar c'_2}}, \quad (12)$$

The operators  $a_i$  and  $a_i^\dagger$  satisfy the undeformed bosonic algebra

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0, \quad [a_i, a_j^\dagger] = i\delta_{ij}.$$

The above undeformed bosonic commutation relation  $[a_i, a_j] = 0$  is automatically satisfied, so in Eqs. (12) the parameter  $c'_2$  is not determined.

If NCQM is a realistic physics, the Hamiltonian of a system reformulated in terms of deformed variables, like the Hamiltonian (4) of the quantum gravitational well in terms of deformed variables  $\hat{x}_i$  and  $\hat{p}_i$ , should have the same representation as the one in terms of

undeformed variables. At the level of annihilation and creation operators in order to guarantee that the Hamiltonian reformulated in terms of  $\hat{a}_i$  and  $\hat{a}_i^\dagger$  has the same representation as the one formulated in terms of  $a_i$  and  $a_i^\dagger$ , the deformed  $\hat{a}_i$  and  $\hat{a}_i^\dagger$  should have the same structure as the undeformed  $a_i$  and  $a_i^\dagger$ . Therefore, the parameters  $c'_1$  and  $c'_2$  in Eqs. (12) should be the same ones  $c_1$  and  $c_2$  in Eqs. (5):

$$c'_1 = c_1, \quad c'_2 = c_2. \quad (13)$$

Inserting Eqs. (2) into Eqs. (5), and using Eqs. (12) and (13), we obtain

$$\hat{a}_i = \xi \left( a_i + \frac{i}{2\hbar} \sqrt{\theta\eta} \epsilon_{ij} a_j \right), \quad \hat{a}_i^\dagger = \xi \left( a_i^\dagger - \frac{i}{2\hbar} \sqrt{\theta\eta} \epsilon_{ij} a_j^\dagger \right). \quad (14)$$

All the deformed bosonic commutation relations in (8) are satisfied by Eqs. (14); Specially,  $[\hat{a}_1, \hat{a}_1^\dagger] = [\hat{a}_2, \hat{a}_2^\dagger] = 1$  are maintained.

It is worth noting that the scaling factor  $\xi$  guarantees consistency of the framework, that is, Eqs. (1)-(3), (6)-(9) and (12)-(14) are consistent each other.

The determinant  $\mathcal{R}'_s$  of the transformation matrix  $R'_s$  between  $(\hat{a}_1, \hat{a}_2, \hat{a}_1^\dagger, \hat{a}_2^\dagger)$  and  $(a_1, a_2, a_1^\dagger, a_2^\dagger)$  in Eqs. (14) is also singular at  $\theta\eta = 4\hbar^2$ . It means that  $(\hat{a}_i, \hat{a}_i^\dagger)$  and  $(a_i, a_i^\dagger)$  are not completely equivalent in physics.

*The tilde system* - In the tilde Fock space, as in commutative space, all calculation is the same, thus the formalism of the deformed Bosonic symmetry which restricts the states under the permutation of identical particles in multi - particle systems can be similarly developed.

Now we consider tilde phase space variables. Using Eqs. (7) and the definition (10) of  $\tilde{a}_i$  we rewrite  $\tilde{a}_i$  as  $\sqrt{\alpha_1} \tilde{a}_1 = \left(\frac{\eta}{4\theta\hbar^2}\right)^{1/4} \left(\tilde{x} + i\sqrt{\frac{\theta\hbar^2}{\eta}} \tilde{p}^\dagger\right)$ ,  $\sqrt{\alpha_2} \tilde{a}_2 = \left(\frac{\eta}{4\theta\hbar^2}\right)^{1/4} \left(\tilde{x}^\dagger + i\sqrt{\frac{\theta\hbar^2}{\eta}} \tilde{p}\right)$ . Where the tilde coordinate and momentum  $(\tilde{x}, \tilde{p})$  are related to  $(\hat{x}, \hat{p})$  by  $\tilde{x} = \frac{1}{\sqrt{2}}(\hat{x}_1 + i\hat{x}_2)$ ,  $\tilde{p} = \frac{1}{\sqrt{2}}(\hat{p}_1 - i\hat{p}_2)$ . The tilde phase variables  $(\tilde{x}, \tilde{p})$  satisfy the following commutation relations:  $[\tilde{x}, \tilde{x}^\dagger] = \xi^2\theta$ ,  $[\tilde{p}, \tilde{p}^\dagger] = -\xi^2\eta$ ,  $[\tilde{x}, \tilde{p}] = [\tilde{x}^\dagger, \tilde{p}^\dagger] = i\hbar$ ,  $[\tilde{x}, \tilde{p}^\dagger] = [\tilde{x}^\dagger, \tilde{p}] = 0$ .

The Hamiltonian  $\hat{H}(\hat{x}, \hat{p}) = \frac{1}{2\mu} \hat{p}_i \hat{p}_i + V(\hat{x}_i)$  with potential  $V(\hat{x}_i)$  in the hat system is rewritten as  $\hat{H}(\hat{x}, \hat{p}) = \tilde{H}(\tilde{x}, \tilde{x}^\dagger, \tilde{p}, \tilde{p}^\dagger) = \frac{1}{2\mu} (\tilde{p} \tilde{p}^\dagger + \tilde{p}^\dagger \tilde{p}) + \tilde{V}(\tilde{x}, \tilde{x}^\dagger)$  in the tilde system. In some cases calculations in the tilde system are simpler than ones in the hat system.

Basis vectors of the tilde Fock space are the common eigen vectors of commutative tilde number operators, so the tilde Fock space is called as the commutative Fock space.

Different from it, Ref. [15] also investigated the structure of a noncommutative Fock space, and obtained eigenvectors of several pairs of commuting hermitian operators which can serve as basis vectors in the noncommutative Fock space.

**Constraint on Quantum Gravitational Well** - In the constrained condition (11) the direct proportional coefficient  $K$  is not determined. For the general case the determination of  $K$  or  $c_2$  based on fundamental principles is an open problem at present. Eqs. (13) represent the condition that the Hamiltonian reformulated in terms of the deformed annihilation and creation operators has the same structure as the one formulated in terms of the undeformed annihilation and creation operators. It shows that the determination of  $c_2$  or  $K$  is related to the determination of the coefficient  $c'_2$  of the undeformed annihilation operator  $a_i$ .

Up to now the harmonic oscillator is the only example for which the explicit representation of  $c'_2$  is known. At the level of ordinary quantum mechanics the dimensional analysis works for the determination of  $c'_2$  up to a dimensionless constant. The dimension of  $c'_2$  in Eqs. (12) is *time/mass*.

The special feature of a harmonic oscillator is that in any state the expectation of the kinetic energy equals to the one of the potential energy. This fixes the dimensionless constant  $\gamma$ . The characteristic parameters in the Hamiltonian of a harmonic oscillator are the mass  $\mu$ , frequency  $\omega$  and  $\hbar$ . The unique product of  $\mu^{t_1}$ ,  $\omega^{t_2}$  and  $\hbar^{t_3}$  possessing the dimension *time/mass* is  $\mu^{-1}\omega^{-1}$ . So one obtains  $c'_2 = \gamma/\mu\omega$ . The position  $x_i$  and momentum  $p_i$  are, respectively, represented by  $a_i$  and  $a_i^\dagger$  as

$$x_i = \sqrt{\frac{\gamma\hbar}{2\mu\omega}} (a_i + a_i^\dagger), \quad p_i = -i\sqrt{\frac{\hbar\mu\omega}{2\gamma}} (a_i - a_i^\dagger).$$

In the vacuum state  $|0\rangle$  the expectations of the kinetic and the potential energy, respectively, read

$$\overline{E_k} = \langle 0 | \frac{1}{2\mu} p_i^2 | 0 \rangle = \frac{\hbar\omega}{4\gamma}, \quad \overline{E_p} = \langle 0 | \frac{1}{2}\mu\omega^2 x_i^2 | 0 \rangle = \frac{\gamma\hbar\omega}{4}.$$

The condition of  $\overline{E_k} = \overline{E_p}$  leads to  $\gamma = \pm 1$ . Because of  $\overline{E_k} \geq 0$ , the only solution is  $\gamma = 1$ .

The second example is a plain electromagnetic wave with a single mode of frequency  $\omega$ . Its characteristic parameters are the frequency  $\omega$ , the fundamental constant  $\hbar$  and  $c$  (the

speed of light in vacuum). The constraint on it is

$$\eta = \kappa \hbar^2 \omega^4 c^{-4} \theta. \quad (15)$$

Here the coefficient  $\kappa$  is an arbitrary dimensionless constant.

Now we consider the quantum gravitational well. Characteristic parameters of the quantum gravitational well are the particle mass  $\mu_n$  and the standard gravitational acceleration  $g$ . Among  $\mu_n$ ,  $g$ , and the fundamental constants  $\hbar$  and  $c$  (the speed of light in vacuum) there are four possible combinations to give the right dimension of  $c'_2$ :  $(\mu_n, g, \hbar)$ ,  $(\mu_n, g, c)$ ,  $(\mu_n, \hbar, c)$  or  $(g, \hbar, c)$ . The deformed Hamiltonian (4) of the quantum gravitational well includes parameters  $\mu_n$ ,  $g$  and  $\hbar$ . In order to obtain a consistent representation of the corresponding Hamiltonian formulated in terms of  $a_i$  and  $a_i^\dagger$ , in the above we should choose the first combination. This gives  $c'_2 = \gamma \left( \frac{\hbar}{\mu_n^4 g^2} \right)^{1/3}$ . From Eqs. (11), (13) and (??) it follows that the constraint on the quantum gravitational well reads

$$\eta = \zeta \left( \frac{\mu_n^4 g^2}{\hbar} \right)^{2/3} \theta, \quad (16)$$

In the above the coefficient  $\zeta$  is an arbitrary dimensionless constant.

The method of determining  $\gamma$  for the harmonic oscillator can not apply to the plain electromagnetic wave and the quantum gravitational well.

We summarize the following points to conclude the paper. (i) In the case of both position - position and momentum - momentum noncommuting Bose - Einstein statistics is guaranteed by the deformed Heisenberg - Weyl algebra itself, independent of dynamics. (ii) A special feature of a dynamical system is represented by a constrained condition. A general feature of such a constraint for any system is a direct proportionality between non-commutative parameters with a coefficient depending on characteristic parameters of the system under study. In the context of non-relativistic quantum mechanics the dimensional analysis can determine the proportional coefficient up to a dimensionless constant. For the general case how to determine such a dimensionless constant is an open problem at present. (iii) The discovery of the deformed correlated bosonic commutation relation (9) makes the deformed bosonic algebra (8) constituting a complete and closed algebra. In literature one is not aware of this correlated bosonic commutation relation. This means that in literature

the deformed bosonic algebra forms a closed algebra, but does not form a complete one. (iv) The deformed annihilation and creation operators  $\hat{a}_i$  and  $\hat{a}_i^\dagger$  are represented by the undeformed ones  $a_i$  and  $a_i^\dagger$  via the linear transformation (14), which maintains the bosonic commutation relations, including  $[\hat{a}_1, \hat{a}_1^\dagger] = [\hat{a}_2, \hat{a}_2^\dagger] = 1$ .

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